

# BASIC SKILLS VERSUS CONCEPTUAL UNDERSTANDING

## *A Bogus Dichotomy in Mathematics Education*

BY H. WU

EDUCATION SEEMS to be plagued by false dichotomies. Until recently, when research and common sense gained the upper hand, the debate over how to teach beginning reading was characterized by many as “phonics vs. meaning.” It turns out that, rather than a dichotomy, there is an inseparable connection between decoding—what one might call the skills part of reading—and comprehension. Fluent decoding, which for most children is best ensured by the direct and systematic teaching of phonics and lots of practice reading, is an indispensable condition of comprehension.

“Facts vs. higher order thinking” is another example of a false choice that we often encounter these days, as if thinking of any sort—high or low—could exist outside of content knowledge. In mathematics education, this debate takes the form of “basic skills or conceptual understanding.” This bogus dichotomy would seem to arise from a common misconception of mathematics held by a segment of the public and the education community: that the demand for precision and fluency in the execution of basic skills in school mathematics runs counter to the acquisition of conceptual understanding. The truth is that in mathematics, skills and understanding are completely intertwined. In most cases, the precision and fluency in the execution of the skills are the requisite vehicles to convey the conceptual understanding. There is not “conceptual understanding” and “problem-solving skill” on the one hand and “basic skills” on the other. Nor can one ac-

*Hung-Hsi Wu is professor of mathematics at the University of California at Berkeley. He has written numerous articles on mathematics education, and is also the co-author with R.K. Sachs of General Relativity for Mathematicians.*

quire the former without the latter.

It has been said that had Einstein been born at the time of the Stone Age, his genius might have enabled him to invent basic arithmetic but probably not much else. However, because he was born at the end of the 19th century—with all the techniques of advanced physics at his disposal—he created the theory of relativity. And so it is with mathematics. Conceptual advances are invariably built on the bedrock of technique. Without the quadratic formula, for example, the theoretical development of polynomial equations and hence of algebra as a whole would have been very different. The ability to sum a geometric series, something routinely taught in Algebra II, is ultimately responsible for the theory of power series, which lurks inside every calculator. And so on.

The analogue of the same phenomenon in the artistic domain is even more transparent. A violinist who still worries about fingering positions cannot hope to impress with the beauty of tone or the elegance of phrasing, and an opera singer without the requisite high notes would try in vain to stir our souls with searing passion. In good art as in good mathematics, technique and conception go hand in hand.

The desire to achieve understanding in a technical subject such as mathematics while minimizing the component of skills is a most human one. There are situations where efforts to this effect are called for and, indeed, brilliantly executed. One can think of the classics of Courant and Robbins (*What Is Mathematics?*)<sup>1</sup> and Hilbert and Cohn-Vossen (*Geometry and the Imagination*).<sup>2</sup> In the context of school mathematics, however, such a desire cannot be indulged without doing great harm to students' education. There are many reasons. Sometimes a simple skill is absolutely indispensable for the understanding of more sophisti-

cated processes. For example, the familiar long division of one number by another provides the key ingredient to understanding why fractions are repeating decimals. Or, the fact that the arithmetic of ordinary fractions (adding, multiplying, reducing to lowest terms, etc.) develops the necessary pattern for understanding rational algebraic expressions. At other times, it is the *fluency* in executing a basic skill that is essential for further progress in the course of one's mathematics education. The automaticity in putting a skill to use frees up mental energy to focus on the more rigorous demands of a complicated problem. Such is the case with the need to know the multiplication table (for single-digit numbers) before attempting to tackle the standard multiplication algorithm, a fact we will demonstrate in due course. Finally, when a skill is bypassed in favor of a conceptual approach, the resulting conceptual understanding often is too superficial. This happens with almost all current attempts at facilitating the teaching of fractions.<sup>3</sup>

Let us illustrate the last statement with the example of the division of fractions. Recall the familiar method of "invert and multiply":

$$\frac{a}{b} \div \frac{c}{d} = \frac{d}{c} \times \frac{a}{b}$$

Nowadays, "invert and multiply" has become almost synonymous with rote learning. Among recent attempts to inject conceptual understanding into this topic, the following approach is not untypical.<sup>4</sup>

Rather than relying on algorithms, where memorization of rules is the focus, the Mathland approach relies heavily on active thinking. To solve problems such as  $\frac{1}{4} \div \frac{1}{2}$ , students need to be able to verbalize the question: How many halves are there in one-fourth? This kind of fluency enables students to use their own logical and visual thinking skills to really know what the solution ( $\frac{1}{2}$ ) means in relation to the problem. *How many halves are there in  $\frac{1}{4}$ ? There is one-half of  $\frac{1}{2}$  in  $\frac{1}{4}$ .*

Many pictures go with the explanation because it is easy to represent one-half, one-fourth, etc., by squares. Three pages down (p. 132), the "invert and multiply" algorithm is introduced and students are urged to "see if the answers you get by using [the algorithm] match up with answers you got earlier this week.... Allow plenty of time to experiment with the standard algorithm, then ask [students] to choose one problem that they worked with both ways and write about how the two solution methods compare." The problems suggested for practice are all of the type  $\frac{5}{6} \div 16$ ,  $\frac{1}{2} \div \frac{1}{6}$ ,  $3 \div \frac{1}{6}$ , etc. With conceptual understanding thus restored—or so it seems—the mathematical exposition on the division of fractions comes to an end.

If only *simple* fractions such as those given above are involved, the preceding approach emphasizing the visual aspect of division is for the most part adequate. The worm in the apple is the need to deal with division problems when the fractions are not at all simple. For example, what do the above brand of logical and visual thinking skills have to say about  $\frac{2}{97} \div \frac{31}{17}$ ? Nothing, of course. A natural consequence of such an approach is that children develop a sense of extreme insecurity upon the sight of any fraction other than the simplest possible.

It is good to start with simple fractions that children can visualize, and they should do many such problems, until they have a firm grasp of what they are doing when they divide fractions. But we should not make students feel that the only problems they can do are those they can visualize. We should explain to them that of course they cannot draw a picture of  $\frac{2}{97} \div \frac{31}{17}$ ; it is doubtful that anyone can. But this does not mean they cannot do the problem! Or that more complex problems like this one are not essential.

An analogy to addition may be helpful. When children were first learning to add, perhaps they counted out three blocks and then counted out four blocks and joined them to get seven blocks. But we didn't tell them that, when faced with the problem  $1,272 + 846$ , their only choice was to gather up hundreds of blocks or draw hundreds of dots on their paper and count them. Nor did we tell them the problem was too difficult for them or not important. No, we told them there was a *mathematical* route to the answer. And not a "rote, meaningless" one, but a procedure based on simple but sound mathematical principles. And we *taught* it to them.

And so we can do with fractions. From the intuitive to the abstract, and from primitive skills to sophisticated ones, such is the normal progression in mathematics. The way to approach the division of non-simple fractions is not to bypass "invert and multiply," but to confront it. We begin by asking what it means to say a fraction

$$\frac{x}{y}$$

equals

$$\frac{a}{b} \div \frac{c}{d},$$

and realize that perhaps we have not fully come to terms with the meaning of the division of whole numbers. Children are taught, for example, that  $24 \div 3 = 8$  means that if you "divide 24 objects into 3 equal portions, each portion would have 8 objects." However, such a grouping of the 24 objects shows that it is  $8+8+8$ , which is therefore the same as  $3 \times 8$ . So in this case, " $24 \div 3 = 8$ " means exactly that  $24 = 3 \times 8$ . This reasoning turns out to be general, in the sense that if we analyze any other example, say  $80 \div 16 = 5$ , then repeating the preceding reasoning leads to a similar conclusion that it is the same as  $80 = 5 \times 16$ . Along this line, fifth-graders should have no trouble understanding that, in general, for whole numbers  $m$ ,  $n$  and  $k$ , the statement

$$m \div n = k$$

says exactly the same thing as

$$m = n \times k.$$

This then provides an abstract point of view to understand division in terms of multiplication. It is common to express this interpretation of division as "division is the inverse operation of multiplication."

With the new insight at hand, we can now reprise the division of fractions: To the extent that whole numbers and fractions are just "numbers," they must share the same properties in terms of the basic operations

such as multiplication or division. Thus looking at each fraction as a number and imitating the case of whole numbers, we see that the division

$$\frac{a}{b} \div \frac{c}{d} = \frac{x}{y}$$

ought to mean

$$\frac{a}{b} = \frac{c}{d} \times \frac{x}{y}.$$

This is then how we want to *define* the division of fractions. Multiplying both sides by

$$\frac{d}{c}$$

immediately leads to

$$\frac{x}{y} = \frac{d}{c} \times \frac{a}{b}.$$

In other words,

$$\frac{a}{b} \div \frac{c}{d} = \frac{d}{c} \times \frac{a}{b}.$$

Thus the method of “invert and multiply” is a result of a *deeper understanding* of fractions than that embodied in the naive logical and visual thinking skills above. We see clearly the concordance of skills and understanding in this instance.

There is at present a desire in a large segment of the education community to achieve understanding of fractions—the bugbear of elementary mathematics education—by avoiding the traditional skills and by restricting attention only to very simple fractions and a naive visual reasoning of the type described above. While the intention is laudable, the inevitable net result is that skills and understanding both are given short shrift. The following passage is another example that sets forth such an agenda:<sup>5</sup>

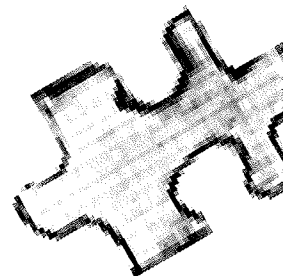
The mastery of a small number of basic facts with common fractions (e.g.,  $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ ;  $\frac{3}{4} + \frac{1}{2} = 1\frac{1}{4}$  and  $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ )... contributes to students' readiness to learn estimation and for concept development and problem solving. This proficiency in the addition, subtraction, and multiplication of fractions and mixed numbers should be limited to those with simple denominators that can be visualized concretely and pictorially and are apt to occur in real-world settings; such computation promotes conceptual understanding of the operations. This is not to suggest however that valuable time should be devoted to exercises like  $\frac{17}{24} + \frac{5}{18}$  or  $5\frac{3}{4} \times 4\frac{1}{4}$ , which are much harder to visualize and unlikely to occur in real-life situations. Division of fractions should be approached conceptually.

Without going into details (which are not unlike those related to the division of fractions), it is again the case that if students only have enough understanding of fractions to do simple operations such as  $\frac{1}{4} + \frac{1}{4}$ ,  $\frac{3}{4} + \frac{1}{2}$ ,  $\frac{1}{2} \times \frac{1}{2}$ , etc., but nothing else, then this understanding is fragile and defective. In this context, it may be worthwhile to point out, in a different way, how the good intention of promoting understanding by suppressing skills can ultimately diminish students' understanding. Both examples of computations,  $\frac{17}{24} + \frac{5}{18}$  and  $5\frac{3}{4} \times 4\frac{1}{4}$ , which students are advised to avoid, are in fact extremely simple to perform. For example, if students have a firm grasp of the distributive law, then:

$$5\frac{3}{4} \times 4\frac{1}{4} = (5 + \frac{3}{4}) \times (4 + \frac{1}{4}) = (5 \times 4) + (5 \times \frac{1}{4}) + (\frac{3}{4} \times 4) + (\frac{3}{4} \times \frac{1}{4}).$$

Because the resulting multiplications and additions on the right are easy to do by any standard, the original computation is also accomplished therewith. The ex-

We should not make students feel that the only problems they can do are those they can visualize.



hortation not to do this computation—although well-intentioned—ends up slighting a very important weapon in students' conceptual arsenal: the distributive law.

There is yet another reason why division of fractions should not be limited to only those problems that students can visualize, which is apparently what the passage quoted above means when it says “approached conceptually.” If students are not fed a steady diet of increasing abstraction, how can they hope to cope with algebra a year or two later? The “algebra for all” battle cry will be an empty promise unless it is backed up by an insistence on elevating education in grades 5 to 7 to periodic heights of abstraction.

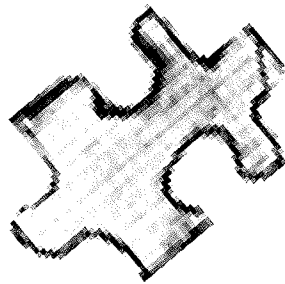
LET US now take up the issue of the teaching of the standard algorithms in elementary school, where the confrontation of skills vs. understanding is most intense. We are told that these algorithms are by their very nature nothing more than rote, meaningless mathematical maneuvers.

Indeed, the very mention of the teaching of standard algorithms causes open hostility in some mathematics education circles. In a recent article, the president of the southern section of the California Mathematics Council put forth the view that the explicit presence of algorithms in the new California Mathematics Standards is nothing less than an advocacy for knowledge to be taken “as a collection of bits or facts to be learned by memorization and impressed upon the child from the outside,” which then results in children trying to “mechanically memorize meaningless facts and skills.”<sup>6</sup>

This view echoes one that is held by many educators, among them Constance Kamii. Kamii is generally acknowledged to be a leading advocate of this point of view. A much quoted recent article co-authored by Kamii and Ann Dominick is provocatively entitled “The Harmful Effects of Algorithms in Grades 1-4.”<sup>7</sup> Its main thesis is this:

Algorithms not only are not helpful in learning arith-

Why not consider the alternative approach of teaching these algorithms *properly* before advocating their banishment from classrooms?



metic, but also hinder children's development of numerical reasoning....

We have two reasons for saying that algorithms are harmful: (1) They encourage children to give up their own thinking, and (2) they "unteach" place value, thereby preventing children from developing number sense....

The persisting difficulty [with standard algorithms] lay in the column-by-column, single-digit approach that prevents children from thinking about multidigit numbers.

This then brings us to an impasse, according to Kamii and Dominick: Children can have conceptual understanding of numbers without learning algorithms, or they become mathematical error-prone robots. Which do we prefer? Invoking Piaget's constructivism [*sic*], Kamii and Dominick recommend that

Children in the primary grades should be able to invent their own arithmetic without the instruction they are now receiving from textbooks and workbooks.

We are thus led to believe that there is no way to teach a simple addition such as  $89 + 34$  (a problem Kamii and Dominick consider) using the standard algorithm except by ramming it down children's throats. Could these authors be unaware of the fact that the addition algorithm, like all other standard algorithms, contains mathematical reasoning that would ultimately enhance children's understanding of our decimal number system? Why not consider the alternative approach of teaching these algorithms *properly* before advocating their banishment from classrooms? Let us see what we can do with the addition algorithm in the special case of  $89 + 34$ .

In a third-grade class, say, let us assume that the chil-

dren already know how to add single-digit numbers fluently. To teach them the addition of  $89 + 34$ , one may begin with a simpler problem:  $59 + 34$ . This is because  $59 + 34$  would avoid any mention of the hundreds digit. Now, one must emphasize at all times that  $59$  is  $50 + 9$  and  $34$  is  $30 + 4$ . So  $59 + 34$  can be added separately in this way:

$$\begin{array}{r} 50 + 9 \\ 30 + 4 \quad (+) \\ \hline 80 + 13 \end{array} \quad (1)$$

Because each "vertical" addition involves only single digits, the individual steps should offer no difficulty to children. Now add  $13$  to  $80$  to get  $93$ ; again this should present no difficulty, because the children can repeat the above process if necessary:

$$\begin{array}{r} 80 \\ 10 + 3 \quad (+) \\ 90 + 3 \end{array} \quad (2)$$

Give several such problems to allow the children to practice addition in this long-winded manner. Because they understand this simple skill, such extended practice to perfect the skill is both necessary and *desirable*. After the students have become thoroughly familiar with the method, point out to them that what they have been doing each time is to add the ones digits *separately*, and then the tens digits *separately*:  $9 + 4$  and  $5 + 3$  in step (1), and  $8 + 1$  and  $0 + 3$  in step (2). Let them do a few more such additions and take note of this fact each time. Allow some time for this idea to sink in before introducing them to the first simplification: Building on the newly acquired idea of adding the digits in different "places" separately, point out to them that they could save some writing in step (2) because they can simply line up the ones and tens digits vertically and directly add since the ones digit in  $80$  would always be  $0$ :

$$\begin{array}{r} 80 \\ 13 \quad (+) \\ \hline 93 \end{array} \quad (3)$$

Again give the students time to get used to this idea. Make them do many practice problems of this type:  $40 + 12$ ,  $60 + 18$ , etc.

Children welcome any suggestions that save labor. It is therefore time to introduce another one. When they can do step (2) in the format of step (3) *fluently*, tell them that in fact they could combine steps (1) and (2) into one step by bringing down the "13" to the next line and add as in step (3):

$$\begin{array}{r} 50 + 9 \\ 30 + 4 \quad (+) \\ \hline 80 + 13 \\ 13 \quad (+) \\ \hline 93 \end{array}$$

With a little bit more practice, the children can simplify the writing even further:

$$\begin{array}{r} (50 + 30) \Rightarrow \begin{array}{r} 59 \\ 34 \quad (+) \\ \hline 80 \end{array} \\ (9 + 4) \Rightarrow \begin{array}{r} 13 \quad (+) \\ \hline 93 \end{array} \end{array}$$

(Students need not write down the two lefthand columns consisting of  $[50 + 30]$ ,  $[9 + 4]$ , and the long right arrows; these only serve as instructional reminders.) The final *coup de grâce*, to be administered only when the children are already secure in all the preceding simpler addition activities, is to point out a *shorthand method* of writing the preceding step: Slip the tens digit "1" of the "13" under 34 to keep track of the addition of the ones digit. So:

$$\begin{array}{r} 59 \\ 34 \\ \hline 1 \quad (+ \\ 93 \end{array}$$

This then is the standard addition algorithm. It should be plain to the children (even if they may not be able to articulate it) that this is an efficient compression of a valuable piece of *mathematical reasoning* into a compact shorthand. They would appreciate this efficiency, let it be noted, only if they have meticulously gone through the laborious process of steps (1) to (3) above. Because young minds are flexible and discerning, the children will learn the algorithm logically without being pressured "from the outside" to "mechanically memorize meaningless facts and skills" while "giving up their own thinking." On the contrary, they will learn how to reason effectively, and the whole experience will stand them in good stead in their later work.

The next step is of course to go back to the original problem of  $89 + 34$ , but the introduction of the hundreds digit in  $80 + 30$  should now present no real difficulty since the simpler case has been firmly mastered.

It may be useful to elaborate on the idea that the standard algorithm presented above captures a valuable piece of mathematical reasoning that enhances students' understanding of numbers. We can see this more clearly by making explicit the underlying mathematics. The fact that  $59 + 34$  can be added as in step (1) makes implicit use of the commutative law and associative law of addition:

$$\begin{aligned} 59 + 34 &= (50 + 9) + (30 + 4) \\ &= [(50 + 9) + 30] + 4 && \text{(assoc. law)} \\ &= [50 + (9 + 30)] + 4 && \text{(assoc. law)} \\ &= [50 + (30 + 9)] + 4 && \text{(comm. law)} \\ &= [(50 + 30) + 9] + 4 && \text{(assoc. law)} \\ &= (50 + 30) + (9 + 4) && \text{(assoc. law)} \end{aligned}$$

Without entering into the tedious details, one need only point out that both laws are also used in all subsequent arguments. Third-graders should not be saddled with this kind of formalism, of course, but teachers should be aware of it if only to gain the confidence that teaching the standard algorithm does not "encourage children to give up their own thinking." Teachers will also need this knowledge to explain it to their students should the need arise.

Children always respond to reason when it is carefully explained to them. The day will come when teachers are capable of explaining these time-honored algorithms in this logical manner. In the meantime, let us be constructive and concentrate on the needed professional development of teachers rather than spread the destructive theory about the harm these algorithms inflict upon children.

To drive home the point that the standard algo-

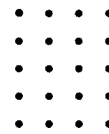
ritms embody conceptual understanding, let us conclude with an examination of the multiplication algorithm as taught to, say, fourth-graders. We assume they are fluent in single-digit multiplications. Consider the problem of  $268 \times 43$ . A new element now appears in the form of the distributive law. Because this law is so basic and because fourth-graders are sufficiently mature to understand it, the law should be explained to them: For any number  $a, b, c$ :

$$a(b + c) = ab + ac.$$

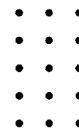
Henceforth, we will write  $a \times b$  as  $ab$  for simplicity. Because multiplication is commutative, this also implies:

$$(b + c)a = ba + ca.$$

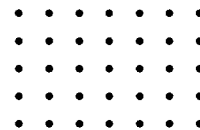
This law can be made plausible using rectangular arrays of dots. For example,  $5 \times 4$  is represented by a five-row and four-column collection of dots.



Similarly  $5 \times 3$  is represented by the dots in:



Hence  $(5 \times 4) + (5 \times 3)$  is represented by the dots obtained from putting the two sets of dots side by side:



But this is a rectangular array of dots with 5 rows and  $(4 + 3)$  columns, so it represents  $5 \times (4 + 3)$ , thereby verifying the distributive law in this special case. The general case is entirely similar.

In the following, we shall call attention to the distributive law each time it is used, but will use the commutative laws and associative laws without mention. To compute  $268 \times 43$ , we shall appeal to the higher order thinking skill of breaking complicated tasks down to simple ones by breaking it down to two simpler multiplication problems. Because 3 and 4 are the two digits of 43, we first compute  $268 \times 3$  and  $268 \times 4$  separately. Because  $268 = 200 + 60 + 8$  (as usual, students need to be reminded of this fact), the distributive law gives:

$$268 \times 3 = (3 \times 200) + (3 \times 60) + (3 \times 8).$$

Because students know how to multiply single-digit numbers, this equals

$$268 \times 3 = 600 + (180) + (24). \quad (4)$$

Because the 180 above comes from the tens digit, we can "split off" the 100 from  $180 = 100 + 80$  and combine it with 600:

$$268 \times 3 = (600 + 100) + (80) + (24). \quad (5)$$

Similarly,  $24 = 20 + 4$ , and we can combine the 20 with the 80 in the tens digit:

$$268 \times 3 = (600 + 100) + (80 + 20) + 4. \quad (6)$$

But now the  $(80 + 20)$  in the tens digit is equal to 100, and we can again combine it with the  $(600 + 100)$  in the hundreds digit. Thus

$$268 \times 3 = (600 + 100 + 100) + 0 + 4 = 800 + 0 + 4 = 804.$$

A few more experiences with working from left to right would tell us that we are likely to waste a little time by so doing because of the frequent need to backtrack to fix a certain digit as in step (6). (There it was the tens digit.) Thus *from experience*, we learn to *work from right to left in order to save time*. This is the reason to work from right to left, but let students find out for themselves by working through several such problems. Therefore, we now redo the above, from right to left, as follows. Start with step (4) again,  $24 = 20 + 4$ , so

$$268 \times 3 = 600 + (180) + (20 + 4) = 600 + (180 + 20) + 4. \quad (7)$$

Next,  $180 + 20 = 200$ , which can be combined with 600:

$$268 \times 3 = (600 + 200) + 0 + 4 = 800 + 0 + 4 = 804. \quad (8)$$

The numbers that were carried from the ones digit to the tens digit in step (7) and from the tens digit to the hundreds digit in step (8) can be recorded by a shorthand method, and this is the standard algorithm for the multiplication of any number by a single-digit number:

$$\begin{array}{r} 268 \\ 3 \quad (\times) \\ \hline 804 \end{array}$$

In exactly the same way, we see that  $268 \times 4 = 1,072$  via the standard algorithm:

$$\begin{array}{r} 268 \\ 4 \quad (\times) \\ \hline 1,072 \end{array}$$

Incidentally, this implies that:

$$268 \times 40 = 10,720. \quad (9)$$

Now we put the pieces together using the distributive law:

$$268 \times 43 = 268 \times (40 + 3) = (268 \times 40) + (268 \times 3).$$

Using steps (8) and (9), we obtain:

$$268 \times 43 = 10,720 + 804 = 11,524.$$

In retrospect, we see that the single-digit approach to this two-digit multiplication problem (that of multiplying by 43) results from *heeding the call of the indispensable mathematical principle to always break down a complicated problem into simple components*. The *correct* way to think about multidigit multiplication is therefore to regard it as nothing more than a sequence of single-digit multiplications. Let children

learn this fundamental fact from day one.

Now to convert the preceding to algorithmic form, it is traditional to use the commutative law of addition to rewrite it as:

$$268 \times 43 = 804 + 10,720 = 11,524$$

so that we have:

$$\begin{array}{r} 268 \\ 43 \quad (\times) \\ \hline 804 \\ 10720 \quad (+) \\ \hline 11524 \end{array} \quad (10)$$

A final touch-up: We see from step (9) that the "0" at the end of 10,720 comes from the "0" of 40, and is the result of 268 being multiplied by the tens digit (4 in this case). Thus this "0" can be taken for granted and will therefore be omitted in the next-to-bottom row of step (10). This accounts for the apparent shift of digits in the next-to-bottom row of the *standard multiplication algorithm*:

$$\begin{array}{r} 268 \\ 43 \quad (\times) \\ \hline 804 \\ 1072 \quad (+) \\ \hline 11524 \end{array} \quad (11)$$

Several observations readily come to mind at this point. The foremost pertains to the clear demonstration of the unity of skills and understanding in this derivation. For example, fluency with single-digit multiplication allows us to take for granted  $268 \times 40$  and  $268 \times 3$  and focus instead on the mathematical ideas leading up to step (10). Another observation is to underscore yet again the central role played by the distributive law, while noting (of course) that the commutative law and associative law also have been used implicitly. For example, in going from step (4) to step (5), we have used the associative law of addition because:  $600 + 180 = 600 + (100 + 80) = (600 + 100) + 80$ . A third observation is that this derivation is nothing if not about place value. The passage from (10) to (11), for example, explains in terms of place value why the digits of the middle two rows have that particular vertical alignment. In what way then does learning the standard algorithms "unlearn" place value?

Finally, we call attention to the breathtaking simplicity of the multiplication algorithm itself despite the tediousness of its derivation. The conceptual understanding hidden in the algorithm is the kind that students eventually need in order to prepare for algebra. In short, this algorithm is a shining example of elementary mathematics at its finest and is fully deserving to be learned by every student. If there is any so-called harmful effect in learning the algorithms, it could only be because they are not taught properly. In Chapter 2 of her pathbreaking book, *Knowing and Teaching Elementary Mathematics*,<sup>8</sup> Liping Ma gives a more refined discussion of why rote learning might take place in the context of multidigit multiplication: It does so when the teacher does not possess a deep enough understanding of the underlying mathematics to explain it well. The problem of rote learning then lies with inadequate professional development and not with the

algorithm. This is exactly the kind of scholarship we need in order to assist our teachers and to move mathematics education forward.

We have given several examples to show that deep understanding of mathematics ultimately lies within the skills. It remains to make a passing comment on the idea of skipping the standard algorithms by asking children to invent their own algorithms instead. The justification is that inventing algorithms promotes conceptual understanding. What is left unsaid is that when a child makes up an algorithm, the act raises two immediate concerns: One is whether the algorithm is correct, and the other is whether it is applicable under all circumstances. In short: correctness and generality. In a class of, say, 30 students, asking the teacher to carefully check 30 new algorithms periodically is a Herculean task. More likely than not, some incorrect algorithms would slip through, and these children would come out of this encounter with mathematics with no understanding at all. Such a potentially harmful effect should have been brought into the open in the advocacy of invented algorithms, but it seems not to have been done. As far as generality is concerned, this aspect of the standard algorithms—the fact that they are applicable under *all* circumstances—seems also to have been neglected in educational discussions. For example, although there are shortcuts to compute special products such as  $97 \times 103$  faster than the standard algorithm, these shortcuts would be of no help at all in a different setting. With each invented algorithm, then, the responsibility of checking its generality again falls on the teacher. Are those who are telling teachers to encourage invented algorithms in their classrooms aware of this heavy burden?

As Euclid told King Ptolemy in the fourth century, B.C., there is no royal road to geometry. Neither is there a royal road to conceptual understanding. Let us teach our children mathematics the honest way by teaching both skills and understanding.  $\square$

## REFERENCES

- <sup>1</sup>Courant, R. & Robbins, H. (1996) *What is mathematics?* Second ed., revised by Ian Stewart. New York, N.Y.: Oxford University Press.
- <sup>2</sup>Hilbert, D. & Cohn-Vossen, S. (1952) *Geometry and the Imagination*. Providence, R.I.: American Mathematical Society.
- <sup>3</sup>The teaching of fractions is the first major bottleneck in school mathematics, and yet it has remained problematic. A major overhaul is needed. Cf. Wu, H. (1998). *Teaching fractions in elementary school: A manual for teachers*. [www.math.berkeley.edu/~wu](http://www.math.berkeley.edu/~wu)
- <sup>4</sup>*Mathland*, (1998) Grade 6. Mountain View, Calif.: Creative Publications. p. 129.
- <sup>5</sup>*Curriculum and Evaluation Standards for School Mathematics*, (1989) Reston, Va.: National Council of Teachers of Mathematics. p. 96.
- <sup>6</sup>Lee, Ellen. (1999) Southern Section Report. *The California Mathematics Council Communicator* 24: 8-9, 11.
- <sup>7</sup>Kamii, C. & Dominick, A. (1998) The harmful effects of algorithms in grades 1-4. In: Kenny, M.J. & Morrow, L.J. (eds). *The Teaching and Learning of Algorithms in School Mathematics*. Reston, Va.: National Council of Teachers of Mathematics.
- <sup>8</sup>Ma, Liping. (1999) *Knowing and Teaching Elementary Mathematics*. Mahwah, N.J.: Lawrence Erlbaum Associates.